

TWO-DIMENSIONAL KNOTS AND REPRESENTATIONS OF HYPERBOLIC GROUPS

BORIS APANASOV[†]

ABSTRACT We describe relations between hyperbolic geometry and knots of codimension two or, more exactly, between varieties of conjugacy classes of discrete faithful representations of the fundamental groups of hyperbolic n -manifolds M into $SO^\circ(n+2, 1)$ and $(n-1)$ -dimensional knots in the $(n+1)$ -sphere S^{n+1} carrying conservative dynamics of hyperbolic lattices $\pi_1(M)$. This approach allows us to discover a phenomenon of non-connectedness of these varieties for closed n -manifolds M , $n \geq 3$, with large enough number of disjoint totally geodesic surfaces, to construct quasisymmetric infinitely compounded “Julia” knots $K \subset S^{n+1}$ which are everywhere wild and have recurrent $\pi_1(M)$ -action, and to study circle and 2-plane bundles (with geometric structures) over closed hyperbolic n -manifolds.

1. INTRODUCTION

This paper studies relations between two spaces which, at first glance, have nothing in common. The first space is the space of conformal classes of m -knots K of dimension $m \geq 2$ in the n -sphere S^n , and the second space is the variety of conjugacy classes of discrete faithful representations of the fundamental group $\pi_1(M)$ of a hyperbolic $(m+1)$ -manifold M into $SO(n+1, 1)$, $n \geq m$, in particular the Teichmüller space $\mathcal{T}(M)$ of conformal structures on the manifold M .

Since our knots are at least two-dimensional, those possible relations are not connected to the problem of existence of hyperbolic metrics (of constant negative curvature) on knot complements $S^n \setminus K$. In fact, in contrast to 1-knots complements in S^3 , there are no hyperbolic metrics on n -knots complements in S^{n+2} , $n \geq 2$. It follows from the fact that parabolic ends of hyperbolic $(n+2)$ -manifolds of finite volume are (up to finite cover) products of Euclidean tori and open intervals, that is hyperbolic metrics may exist only in the complements of Euclidean surfaces, see [A1] for example.

Nevertheless, one can apply Tukia’s [Tu] construction of the canonical (quasisymmetric) homeomorphisms of the limit sets of isomorphic geometrically finite hyperbolic groups to establish such a relation. It immediately implies (see [A8]):

There exists a canonical continuous map Φ of the Teichmüller space $\mathcal{T}^n(G)$ of geometrically finite faithful representations of a uniform hyperbolic lattice $G \subset O(m+1, 1)$, $m \geq 2$, to the space $\mathcal{QS}_{m,n}(G)$ of conformal classes of G -equivariant

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quasisymmetric embeddings $S^m \hookrightarrow S^n$. Moreover, this map Φ is at most a (2-1)-map.

In particular, if $n = m + 2$ the space $\mathcal{QS}_{m,n}(G)$ consists of (conformal classes of) m -knots $K \subset S^{m+2}$. We also note that the last statement on (2-1)-maps follows from the Mostow rigidity ([Mo, Ma]) and the Gluck [Gl] theorem (see also [B, LS, Sw]) that two homeomorphisms of the boundary $\partial N(K) \approx S^m \times S^1$ of a regular neighborhood $N(K) \approx S^m \times B^2$ of a m -knot K , $m \geq 2$, are pseudo-isotopic if and only if they are homotopic.

Our main results use the map Φ for a solution of the connectedness problem for the above spaces of m -knots and the Teichmüller spaces associated with a hyperbolic lattice. This solution is based on our construction in the following theorem.

Theorem 3.1. *For a given nontrivial ribbon m -knot $K \subset S^{m+2}$, $m \geq 2$, there exists a discrete faithful representation $\rho : \Gamma \rightarrow \text{Möb}(m+2)$ of a uniform hyperbolic lattice $\Gamma \subset \text{Isom } \mathbb{H}^{m+1}$ such that the Kleinian group $G = \rho\Gamma$ acts ergodically on the everywhere wild quasisymmetric m -knot $K_\infty = \Lambda(G) \subset S^n$ obtained as an infinite compounding of the knot K .*

As a corollary of this theorem, we have

Theorem 5.2. *Let $\Gamma \subset \text{Isom } \mathbb{H}^m$ be any uniform hyperbolic lattice from the above theorem, and $m \geq 3$. Then the varieties $\mathcal{T}^{m+1}(\Gamma)$ and $\mathcal{H}^{m+2}(\Gamma)$ of conformal and hyperbolic structures on Γ (respectively, the classes of discrete faithful representations $\Gamma \rightarrow \text{SO}^\circ(m+2, 1)$) are not connected.*

We should mention that this non-connectedness phenomenon has no relation to non-connectedness of the (smaller) Teichmüller space $\mathcal{T}^m(\Gamma)$ of conformal m -structures on $M = \mathbb{H}^m/\Gamma$, see [A5]. Indeed, the topological obstruction for connectedness of the space $\mathcal{T}^3(M)$ of conformal structures on $M = \mathbb{H}^3/\Gamma$ (nontrivial knotting of the limit 2-sphere $\Lambda(G) \subset S^3$) is not an obstruction for connectedness of $\mathcal{T}^4(\Gamma)$ and $\mathcal{H}^5(\Gamma)$. In fact, the nerve of that knotting $S^2 \hookrightarrow S^3$ is 1-dimensional, and hence the knotted in S^3 2-sphere $\Lambda(G)$ is unknotted in S^m , $m \geq 4$.

Another relation between m -knots and hyperbolic manifolds is based on the Gluck's rigidity of Dehn surgeries on trivial knots of dimension at least 2:

Theorem 6.1. *For a given closed hyperbolic m -manifold M , $m \geq 3$, there are at most two non-equivalent circle (or 2-plane) bundles over M allowing uniformizable conformal structures or complete hyperbolic metrics, respectively, with the development on a $(m-1)$ -knot complement.*

This paper is preserved in the form it was written in 1995, so it may have no references to results published after that time. In conclusion, the author would like to thank Scott Carter and Masahico Saito for helpful conversations.

2. VARIETIES OF REPRESENTATIONS AND GEOMETRIC STRUCTURES

Let M be a given closed hyperbolic m -manifold (orbifold), $m \geq 3$, that is a complete oriented Riemannian manifold with constant sectional curvature -1, and $\pi_1(M) = \Gamma \subset \text{SO}^\circ(m, 1)$ its fundamental group isometrically acting in the hyperbolic m -space \mathbb{H}^m as a uniform lattice, $M = \mathbb{H}^m/\Gamma$. One can consider the variety $\mathcal{P}^n(\Gamma) = \text{Hom}(\Gamma, \text{SO}^\circ(n, 1))$ of all representations of Γ into $\text{SO}^\circ(n, 1)$, $n \geq m$ with

the algebraic convergence topology where the group $\mathrm{SO}^\circ(n, 1)$ acts by conjugations. Inside of the quotient-variety $\mathcal{R}^n(\Gamma)/\mathrm{SO}^\circ(n, 1)$ there are two subvarieties

$$\mathcal{T}^{n-1}(\Gamma) \subset \mathcal{H}^n(\Gamma) \subset \mathcal{R}^n(\Gamma)/\mathrm{SO}^\circ(n, 1) \quad (2.1)$$

which both consist of conjugacy classes of faithful discrete representations, with an additional condition on representations ρ , $[\rho] \in \mathcal{T}^{n-1}(\Gamma)$, that $\rho(\Gamma) \subset \mathrm{SO}^\circ(n, 1)$ are geometrically finite with non-empty discontinuity sets $\Omega(\Gamma) \subset S^{n-1} = \partial\mathbb{H}^n$ (the complements of the limit sets $\Lambda(\Gamma)$, $\Omega(\Gamma) = S^{n-1} \setminus \Lambda(\Gamma)$). Due to the Sullivan's stability theorem (see [Su], [A9, Th.7.2]), the subvariety $\mathcal{T}^{n-1}(\Gamma)$ is open in $\mathcal{H}^n(\Gamma)$.

There is a natural identification [Mo] of the biggest space $\mathcal{H}^n(\Gamma)$ with the space of n -dimensional hyperbolic structures on the group Γ . Such a hyperbolic structure on Γ is determined (up to inner automorphisms of Γ and hyperbolic isometries) by a pair $\{N, \phi\}$, where N is an n -dimensional hyperbolic manifold and $\phi : \Gamma \rightarrow \pi_1(N)$ is an isomorphism. In particular, for a closed surface S_p of genus $p > 1$ and $\Gamma = \pi_1(S_p)$, the space $\mathcal{H}^2(\Gamma)$ is the Teichmüller space $\mathcal{T}(\Gamma)$ of Γ homeomorphic to \mathbb{R}^{6p-6} , and the space $\mathcal{H}^3(\Gamma)$ is isomorphic to the product $\mathcal{T}(\Gamma) \times \mathcal{T}(\Gamma)$.

Similarly, via holonomy, we have a natural identification of the space $\mathcal{T}^{n-1}(\Gamma)$ with the (Teichmüller) space of uniformizable $(n-1)$ -dimensional conformal (=conformally flat) structures on Γ (cf. [A1, A4]). An element of this space is determined by a pair $\{N, \phi\}$, where N is a conformal (=conformally flat) $(n-1)$ -manifold with a non-surjective developing map $d : \tilde{N} \rightarrow S^{n-1}$ of its universal covering space \tilde{N} into $d(\tilde{N}) = \Omega \subset S^{n-1}$ and $\phi : \Gamma \rightarrow \pi_1(N)$ is a monomorphism corresponding to the short exact sequence

$$0 \rightarrow \pi_1(\Omega) \rightarrow \pi_1(N) \rightarrow \Gamma \rightarrow 0. \quad (2.2)$$

Here, due to [Ka] and [KP], the developing map d and the natural projection $\pi : \Omega \rightarrow \Omega/G \cong N$ (where $\Gamma \cong G = d_*(\pi_1(N)) \subset \mathrm{SO}^\circ(n, 1)$) are covering maps which factor the universal projection $\tilde{N} \rightarrow N$. Two pairs $\{N_0, \phi_0\}$ and $\{N_1, \phi_1\}$ determine the same conformal structure on Γ if there is an orientation preserving conformal homeomorphism $f : N_0 \rightarrow N_1$ such that $f_* \circ \phi_0$ and ϕ_1 differ (up to the isotropy subgroup $Z(\rho_0)$ of the inclusion $\rho_0 : \Gamma \rightarrow \mathrm{SO}^\circ(n, 1)$) by an inner automorphism of Γ . (Notice that $f_* : \pi_1(N_0) \rightarrow \pi_1(N_1)$ is only well defined up to inner automorphism of $\pi_1(N_0)$.)

The first surprising results on the varieties $\mathcal{T}^{n-1}(\Gamma)$ and $\mathcal{H}^n(\Gamma)$ (beyond the Mostow rigidity [Mo, Ma]) are that there are large classes of groups Γ for which these varieties are non-trivial ([A2, A1]), but $\mathcal{H}^n(\Gamma)$ is still compact ([T, MS, Mo]). In particular, due to J.Morgan's weak rigidity theorem [Mo], we have (cf. [A4]):

Theorem 2.1. *Let M be a closed oriented hyperbolic m -manifold, $m \geq 3$, with $\pi_1(M) \cong \Gamma \subset \mathrm{SO}^\circ(m, 1)$. Then, for any $n \geq m$, the (Teichmüller) variety $\mathcal{T}^{n-1}(\Gamma)$ has a natural compactification $\overline{\mathcal{T}^{n-1}(\Gamma)}$ such that each of its points corresponds to a faithful discrete representation $\rho : \Gamma \rightarrow \mathrm{SO}^\circ(n, 1)$ from the compact variety $\mathcal{H}^n(\Gamma)$.*

3. KNOTTED m -SPHERES IN THE $(m+2)$ -SPHERE S^{m+2}

An m -knot $K \subset S^{m+2}$ is an embedding $K : S^m \hookrightarrow S^{m+2}$ of the oriented m -sphere into the oriented $(m+2)$ -sphere. Two m -knots K_1 and K_2 in S^{m+2} are called *equivalent* (or of the same type) if there exists an orientation preserving

homeomorphism $f : S^{m+2} \rightarrow S^{m+2}$ such that $fK_1 = K_2$. Obviously, it is an equivalence relation, and we call the equivalence class $[K]$ of a m -knot $K \subset S^{m+2}$ the *knot type* of K . Those m -knots that are equivalent to the natural inclusion $S^m \subset S^{m+2}$ are called *trivial* or *unknotted*.

The simplest examples of nontrivial 2-knots in S^4 can be obtained by using the so-called suspensions and spins of classical knots in S^3 , see [Ar], [Ro, §3J]. Namely, for a classical nontrivial 1-knot $k \subset S^3$, let points a and b lie in disjoint components S_+^4 and S_-^4 of $S^4 \setminus S^3$. Then the join K of the knot k with $\{a\} \cup \{b\}$ is called the *suspension* of the 1-knot k , see Fig. 1. It holds that $\pi_1(S^4 \setminus K) \cong \pi_1(S^3 \setminus k)$, so the obtained 2-knot K is nontrivial.

To construct a spun 2-knot $K \subset S^4$, we consider a classical knot k in the half-space $\overline{\mathbb{R}_+^3} = \{x \in \mathbb{R}^4 : x_4 = 0, x_3 \geq 0\}$, such that the boundary 2-plane $\mathbb{R}^2 = \{x \in \mathbb{R}^4 : x_3 = x_4 = 0\}$ intersects the knot k along a single arc α . Then spinning the complementary arc $\beta = \text{cl}(k \setminus \alpha)$ about the plane \mathbb{R}^2 sweeps out a 2-sphere $K \subset \mathbb{R}^4$. Obviously, $K \cap \overline{\mathbb{R}_+^3}$ consists of a simple loop k_1 . Providing an orientation on k_1 which is coherent to that of the arc $\beta \subset k_1$, we see that k_1 is representing the 1-knot that is obtained as the connected sum $k \# (-k) \subset \mathbb{R}^3$, see Fig. 2.

The 2-sphere $K \subset S^4$ is oriented so that the orientation of the 2-disk $K \cap S_+^4$ is coherent to that of k_1 . So obtained 2-knot $K : S^2 \hookrightarrow S^4$ is called the *spun 2-knot* of a given 1-knot $k \subset S^3$, see Fig. 2. Since every loop in $S^4 \setminus K$ can be continuously deformed in $\mathbb{R}_+^3 \setminus \beta$, we have that

$$\pi_1(S^4 \setminus K) \cong \pi_1(\mathbb{R}_+^3 \setminus \beta) \cong \pi_1(\mathbb{R}^3 \setminus k).$$

We note also that every spun 2-knot $K \subset S^4$ is a *ribbon 2-knot*. Such 2-knots generalize classical ribbon knots in S^3 , see [Ro, BZ]. They can be obtained as follows, see [Suz] and Fig. 3.

Let $S_0 \cup \dots \cup S_m \subset \mathbb{R}^4$ be a trivial 2-link with $(m+1)$ components (which are trivial non-linked 2-knots) and f_1, \dots, f_m are appropriate embeddings of 3-balls, $f_i : [0, 1] \times B^2 \hookrightarrow \mathbb{R}^4$, which make m fusions of the 2-link. Each of the embeddings f_i is such that

$$f_i([0, 1] \times B^2) \cap (S_0 \cup \dots \cup S_m) = f_i(\{0, 1\} \times B^2)$$

has an orientation coherent with that of the 2-link, and the disks $f_i(\{0\} \times B^2)$ and $f_i(\{1\} \times B^2)$ are contained in different components of the link. Then the connected sum of the spheres S_0, \dots, S_m and the sphere $f_i(\partial([0, 1] \times B^2))$ represented by the homological sum

$$(S_0 \cup \dots \cup S_m) + f_i(\partial([0, 1] \times B^2)) = S_0^1 \cup \dots \cup S_{m-1}^1$$

is a trivial 2-link with $(m-1)$ components. Continuing this process of fusions on the link, we finally obtain a 2-knot which is called a *ribbon 2-knot with m fusions*, see Fig. 3.

Similarly, for $m \geq 2$, one can define the above classes of m -knots in $(m+2)$ -sphere S^{m+2} , in particular ribbon m -knots with a given number of fusions.

Now we can describe a connection between m -knots in S^{m+2} , $m \geq 2$, and varieties of discrete representations of hyperbolic lattices $\Gamma \subset \text{Isom } \mathbb{H}^{m+1}$, which is based on the following theorem.

Theorem 3.1. *For a given nontrivial ribbon m -knot $K \subset S^{m+2}$, $m \geq 2$, there exists a discrete faithful representation $\rho : \Gamma \rightarrow \text{Möb}(m+2)$ of a uniform hyperbolic lattice $\Gamma \subset \text{SO}^\circ(m+1, 1) \cong \text{Isom}_+ \mathbb{H}^{m+1}$ such that the Kleinian group $G = \rho\Gamma$ acts ergodically on the everywhere wild m -knot $K_\infty = \Lambda(G) \subset S^n$ obtained as an infinite compounding of the knot K .*

4. BASIC CONSTRUCTION

The proof of the above Theorem 3.1 is based on the author's "block-building method" (see [A3, A5]) and geometrically controlled PL-approximations of smooth ribbon $(n-2)$ -knots $K \subset S^n$, $n \geq 4$. Namely, we may assume that the $(n-2)$ -dimensional spheres $S_0, \dots, S_m \subset S^n$ and the embeddings $f_i : B^{n-1} \approx [0, 1] \times B^{n-2} \hookrightarrow S^n$ in the definition of a given ribbon $(n-2)$ -knot $K \subset S^n$ are taken in the conformal category. That means that all involved spheres are round spheres, and each image $f_i(B^{n-1})$ is contained in the union of finitely many round $(n-1)$ -balls B_j in S^n , $1 \leq j \leq j_i$, such that the boundary spheres of any two adjacent balls intersect each other along a round $(n-3)$ -sphere, See Fig 4.

In other words, the $(n-1)$ -dimensional ribbon $f_i(B^{n-1})$ can be obtained from a flat ribbon in \mathbb{R}^{n-1} , which is the union of round balls, by sequential bendings along $(n-2)$ -planes. We do that by using the well known construction of bending deformations (see [A1, A2, Ko, T, A9]).

Let us assume in addition that, in each round $(n-1)$ -ball B_j in the construction (either a ball from one of the ribbons $f_i(B^{n-1})$ or one of the balls bounded by spheres S_k , $0 \leq k \leq m$), there is a discrete action of a hyperbolic group $G_j \subset \text{Isom } \mathbb{H}^{n-1} = \text{Möb}(B_j)$. Up to isotopy of the $(n-2)$ -knot K and the family Σ of $(n-1)$ -balls B_j , we may assume that the groups G_j have bending hyperbolic $(n-2)$ -planes whose boundaries at infinity ∂B_j are the intersection spheres $\delta_j = \partial B_j \cap \partial B_{j+1}$ for the adjacent balls B_j and B_{j+1} , and that the stabilizers of δ_j in G_j and G_{j+1} coincide. We denote such stabilizers by $\Gamma_j = G_j \cap G_{j+1}$. This property guarantees that the amalgamated free product

$$G = \cdots *_{{\Gamma_{j-1}}} G_j *_{{\Gamma_j}} G_{j+1} *_{{\Gamma_{j+1}}} \cdots \subset \text{Möb}(n) \quad (4.1)$$

is a Kleinian group isomorphic to a hyperbolic uniform lattice $\Gamma \subset \text{Isom } \mathbb{H}^{n+1}$.

As the result of this geometric construction, we have that our ribbon $(n-2)$ -knot $K \subset S^n$ is represented as the union K_0 of $m+1$ disjoint round $(n-1)$ -spheres with $2m$ deleted disjoint round $(n-1)$ -balls on them,

$$(S_0 \cup \dots \cup S_m) \setminus \bigcup_{i=1}^{2m} B_i^{n-1}; \quad B_i^{n-1} \subset S_k, \quad 1 \leq i \leq 2m, \quad 0 \leq k \leq m,$$

and disjoint $(n-2)$ -dimensional cylinders corresponding to the ribbons f_1, \dots, f_m of the knot K . These disjoint cylinders are the unions of spherical $(n-2)$ -dimensional annuli with disjoint interiors which lie on the boundary spheres ∂B_j of the round $(n-1)$ -balls B_j in the construction. Due to the choice of the block-groups $G_j \subset \text{Möb}(B_j) \subset \text{Möb}(n)$, the knot $K_0 \cong K$ lies in the interior of the complement of a fundamental polyhedron $F \subset S^n$ of the product group G in (4.1). Thus $S^n \setminus F$ is a regular neighborhood of K . Furthermore, essentially using bending

deformations (along hypersurfaces corresponding to amalgama-subgroups Γ_j , see [A3, A1]), we can construct a hyperbolic uniform lattice $\Gamma \subset \text{Isom } \mathbb{H}^{n+1}$ which has the same amalgamated free product structure as the group G in (4.1) (and hence, it is isomorphic to G) and conformally acts in one of the $(n-1)$ -dimensional balls B_j . We do that in a way similar to [A3].

Now a direct application of Tukia's [Tu] isomorphism theorem shows that the limit set $\Lambda(G)$ of the group $G \cong \Gamma \subset \text{Isom } \mathbb{H}^{n+1}$ is an $(n-2)$ -knot in S^n . We claim that it is the desired everywhere wild knot $K_\infty \subset S^n$ obtained by the infinite compounding of the knot $K_0 = K$:

$$K_\infty = \dots \# K \# K \# K \# \dots \quad (4.2)$$

Before we go on with the proof, we shall show how this construction works in the case of 2-knots. We illustrate it by a simplest ribbon 2-knot $K \subset \mathbb{R}^4$ obtained by one fusion from two unlinked 2-spheres S_0 and S_1 , see Fig. 3. This knot K is also the spun 2-knot of the classical trefoil knot $k \subset \mathbb{R}^3$, see Fig. 2.

We can take the knot $K \subset S^4$ to be a PL-knot shown in Fig. 5, that is as a ribbon 2-knot obtained by one fusion of two unlinked boundaries of 3-dimensional cubes Q_0 and Q_1 in 3-planes $\mathbb{R}^3 \times \{0\}$, $\mathbb{R}^3 \times \{54\} \subset \mathbb{R}^4$, respectively.

To have better geometric control on the blocks that we are going to use in the construction, we shall use (instead of the spheres S_0 and S_1 in the definition of a ribbon knot) the cubes ∂Q_i with edges parallel to the coordinate axes in \mathbb{R}^4 . As the ribbon $f_1 : B^3 \hookrightarrow \mathbb{R}^4$ we shall also use the union of 3-dimensional (smaller) cubes Q_j , $2 \leq j \leq m$, in \mathbb{R}^4 as it is indicated in Fig. 5.

Here the edges of the equal small cubes Q_j , $j \geq 2$, are parallel to the coordinate axes in \mathbb{R}^4 . Their size shall be later determined by the size of the first two bigger cubes $Q_0 \cong Q_1$. The plus signs in Fig. 5 show the character of intersections in our 3-dimensional projection of K of the boundaries ∂Q_0 and ∂Q_1 with the boundary of 3-dimensional tube which is the union $\bigcup_{2 \leq j \leq m} Q_j$ of small cubes. Later we shall also explain our choice of parallel 3-planes $\mathbb{R}^3 \times \{-27\}$, $\mathbb{R}^3 \times \{0\}$, $\mathbb{R}^3 \times \{54\}$ and $\mathbb{R}^3 \times \{81\}$ in \mathbb{R}^4 which contain some of the cubes Q_j and are orthogonally joined by tubes that are boundaries of the union of the remaining small cubes.

Now we define discrete block-groups G_j associated with the cubes Q_j . Although G_j are isomorphic to hyperbolic isometry groups in \mathbb{H}^3 , it is more convenient to use quasi-Fuchsian deformations of these hyperbolic groups (bendings as in citeA2, A3) so that the obtained groups G_j match the cubes Q_j in the following sense.

Assuming $K = K_0$, we cover the 2-knot $K \subset \bigcup_{0 \leq j \leq m} \partial Q_j$ by a family $\Sigma = \{b_{ji}\}$ of closed round 4-balls b_{ji} whose boundary spheres ∂b_{ji} are orthogonal to K . Namely, in the first step, we take 4-balls b_{ji} centered at the vertices of the cubes Q_j , $0 \leq j \leq m$, whose radii r_{ji} are equal to each other if either $j = 0, 1$ or $2 \leq j \leq m$. One more condition on these radii r_{ji} is that $b_{ji} \cap b_{kl} \neq \emptyset$ only if the centers of the different balls b_{ji} and b_{kl} are the ends of a common 1-edge of one of the cubes. In the latter case, the magnitude of the exterior dihedral angle bounded by the spheres ∂b_{ji} and ∂b_{kl} should equal $\pi/3$. These 4-balls b_{ji} do not cover the entire knot K . On each square 2-side $X \subset \partial Q_j \cap K$, we have uncovered 4-gon bounded by circular arcs. We cover such a 4-gon by five additional 4-balls b_{ji} centered at X and whose boundary spheres ∂b_{ji} intersect (orthogonally) only those previously constructed balls that are centered at the vertices of X . Among these five new balls, the first four sequentially intersect each other with the external dihedral angles $\pi/3$. The

fifth ball is centered at the center of X and (orthogonally) intersects only the last new four balls, see Fig. 6. After that, we still have uncovered those two 2-sides X_0 and X_1 of the big cubes Q_0 and Q_1 that are (orthogonally) joined by the tube $\bigcup_{2 \leq j \leq m} Q_j$. Here we assume that $Q_0 \cap Q_2$ and $Q_1 \cap Q_m$ are small squares centered at the centers of X_0 and X_1 , respectively. Furthermore, we choose *the size* of the cubes Q_j so that Q_j , $j \geq 2$, are unit cubes and the cubes Q_0 and Q_1 have the size which matches the covering family $\{b_{ji}\}$. In fact, the boundary spheres of the four additional balls centered at $\text{int}(X_0)$ orthogonally intersect the corresponding spheres centered at the four vertices of the small cube Q_2 , see Fig. 6.

This completes the construction of the family $\Sigma = \{b_{ji}\}$ of 4-balls covering the knot K . The union of these balls, $\bigcup_{i,j} \text{int} b_{ji} = N(K)$, is a regular neighborhood of the PL-ribbon knot K .

Notice that we can take the size of cubes Q_j , $j \geq 2$, arbitrarily smaller than that of the cubes Q_0 and Q_1 . To do that, we repeat the above process of covering the sides X_0 and X_1 by balls b_{ji} where, instead of the vertices of X_0 (and X_1), we take the centers of the four new small balls. Then each of the annuli in $X_0 \setminus Q_2$ and $X_1 \setminus Q_m$ will be covered by $(4 + 8k)$ additional balls b_{ji} (for sufficiently large integer $k \geq 0$) instead of the above four additional balls corresponding to $k = 0$. This allows us to take the ribbon $f_1 : B^3 \hookrightarrow \mathbb{R}^4$ as thin as we need.

We define a discrete block-group G_j associated with a cube Q_j , $0 \leq j \leq m$, as the group generated by reflections with respect to all spheres ∂b_{ji} , that is, with respect to all spheres ∂b_{ji} that intersect the cube Q_j . Obviously, G_j is discrete because all spherical dihedral angles with edges $\partial b_{ji} \cap \partial b_{jl}$ are either $\pi/3$ or $\pi/2$ (see [A1, Mas] for example). Furthermore, G_j preserves each of (coordinate) 3-planes $\mathbb{R}^3 \subset \mathbb{R}^4$ that contain the cube Q_j . In such a 3-plane \mathbb{R}^3 , the group G_j can be deformed by bendings to a Fuchsian group acting in a 3-ball $B^3 \subset \mathbb{R}^3$. That is why we can consider the groups G_j as discrete subgroups in $\text{Isom } \mathbb{H}^3$, $G_j \cong G'_j \subset \text{Isom } \mathbb{H}^3$.

For any two adjacent cubes Q_j and Q_{j+1} , the groups G_j and G_{j+1} have a common subgroup $\Gamma_j = G_j \cap G_{j+1}$ which is generated by four reflections with respect to the spheres centered at the vertices of the square $Q_j \cap Q_{j+1}$. So we can apply the Maskit combination [Mas, A1] and obtain a Kleinian group $G \subset \text{Möb}(4)$ as the free amalgamated product in (4.1). For the group G , we can take the complement of a regular neighborhood $N(K)$ of the knot K to be a fundamental polyhedron $P \subset S^4$:

$$P = \overline{\mathbb{R}^4} \setminus N(K), \quad N(K) = \bigcup_{i,j} \text{int } b_{ji}. \quad (4.3)$$

We remark that, for each amalgamated free product $G_j *_{\Gamma_j} G_{j+1}$, we can use a bending deformation along the hyperbolic 2-plane H_j whose boundary circle ∂H_j is the limit circle of the amalgama subgroup Γ_j . As a result, we get a new hyperbolic isometry group $G'_j \subset \text{Isom } \mathbb{H}^3$ which is isomorphic to $G_j *_{\Gamma_j} G_{j+1}$. Applying this process m times, we obtain a cocompact discrete group $\Gamma \subset \text{Isom } \mathbb{H}^3$ isomorphic to the group G .

In dimension $n = 4$, there is another (non-algorithmical) way to get such a unique hyperbolic lattice Γ by using the Andreev-Rivin classification of hyperbolic compact polyhedra in \mathbb{H}^3 . Namely, the boundary ∂P of the polyhedron in (4.3) has the combinatorial type of $S^2 \times S^1$ where the 2-sphere S^2 is decomposed into the union of spherical polyhedra. In fact, ∂P is the union of 2-sides each of which is the

annulus on a sphere ∂b_{ji} , i.e. each 3-side is the product of a spherical 2-polygon D_{ji} and the circle S^1 . The dihedral angles between such 3-sides are determined by the corresponding 3-dimensional dihedral angles bounded by 2-spheres $\partial b_{ji} \cap \mathbb{R}^3$ in the corresponding 3-planes $\mathbb{R}^3 \subset \mathbb{R}^4$, so they are either $\pi/3$ or $\pi/2$, and the Andreev-Rivin conditions apply (see [An, Ri]). It follows that the combinatorial type of the 4-polyhedron P determines the combinatorial type of a 3-dimensional compact hyperbolic polyhedron $P' \subset \mathbb{H}^3$, with the same magnitudes of dihedral angles as those for P . Thus the group $\Gamma \subset \text{Isom } \mathbb{H}^3$ generated by reflections in sides of P' is a uniform hyperbolic lattice isomorphic to the group G .

Remark 4.1. The above observation that we can take the ribbon $f_1 : B^3 \hookrightarrow \mathbb{R}^4$ “arbitrarily thin” makes it possible to apply the above block-groups $G_j \subset \text{Möb}(4)$, $G_i \cong G'_j \subset \text{Isom } \mathbb{H}^3$, to represent an arbitrary ribbon 2-knot $K \subset S^4$ as the knot which lie on the boundary of the union of 3-cubes similar the above cubes Q_j .

To finish the proof of Theorem 3.1, we need to show that the knot $K_\infty = \Lambda(G)$ in (4.2) is an everywhere wild $(n-2)$ -knot if the knot K is nontrivial.

Let $\mathfrak{S} = G(\bigcup_{ij} \partial b_{ji})$ be the G -orbit of the boundary spheres ∂b_{ji} of the balls in the covering Σ of the PL-knot K . We can use the word norm $|g|$ of elements $g \in G$ with respect to generators of G , which are reflections in sides of the fundamental polyhedron $P = S^n \setminus N(K)$ in (4.3), to define a partial ordering on \mathfrak{S} . Namely, for two spheres $S_1, S_2 \in \mathfrak{S}$, we say $S_2 \succcurlyeq S_1$ if $\text{int } S_2 \subseteq \text{int } S_1$. It allows us to enumerate the set \mathfrak{S} by a bijection $q : \mathbb{N} \rightarrow \mathfrak{S}$ so that it is compatible with the ordering of \mathfrak{S} , that is the map q^{-1} preserves this partial order. Then we have a nested sequence of compacta,

$$P_0 = P \subset P_1 = P_0 \cup g_1(P_0) \subset \dots \subset P_k = P_{k-1} \cup g_k(P_{k-1}) \subset \dots \subset \Omega(G), \quad (4.4)$$

where the elements $g_i \in G$ are the reflections with respect to i -th spheres $S_i \in \mathfrak{S}$ each of which contains a side of the $(i-1)$ -th polyhedron P_{i-1} .

The complement $S^n \setminus P_i$ of each of the compacta P_i in (4.4) is a regular open neighborhood of an $(n-2)$ -knot K_i which is obtained from the knot K by sequential connected sums:

$$K_0 = K, \quad K_1 = K_0 \# K_0, \dots, K_i = K_{i-1} \# K_{i-1}, \dots \quad (4.5)$$

Since the limit set $\Lambda(G) = S^n \setminus \Omega(G)$ is homeomorphic to the limit set $\Lambda(\Gamma) = S^{n-2}$ (Tukia's [Tu] isomorphism theorem), $\Lambda(G)$ is an embedded $(n-2)$ -sphere in S^n . We denote $\Lambda(G) = K_\infty$ and claim that it is an everywhere wild $(n-2)$ -knot in S^n . Obviously, $K_\infty = \bigcap_i \overline{N(K_i)}$ where, for any i , $N(K_i) = S^n \setminus P_i$ is a regular neighborhood of the knot K_i in (4.5). Due to (4.4), the nested sequence $\{\overline{N(K_i)}\}$ is decreasing to its intersection, K_∞ .

Due to the Alexander duality [Sp] applied to $\Omega(G) = S^n \setminus K_\infty$, we have that $H_1(\Omega(G); \mathbb{Z}) \cong H^{n-2}(S^{n-2}; \mathbb{Z}) = \mathbb{Z}$. Thus we can consider an infinite cyclic covering space $\widehat{\Omega}$ of $\Omega(G)$. Now we are concerned with the integral homology $H_*(\widehat{\Omega}) = H_*(\widehat{\Omega}; \mathbb{Z})$ with Λ -module structure where Λ denotes the ring of finite Laurent polynomials with integer coefficients. Namely, choosing a generator $\tau : \widehat{\Omega} \rightarrow \widehat{\Omega}$ of the deck transformation group of the cyclic covering $\widehat{\Omega} \rightarrow \Omega(G)$, we define the product of an element $x(t) = \sum_{i \in \mathbb{Z}} x_i t^i \in \Lambda$ with an element $y \in H_*(\widehat{\Omega})$ as

$$p(t) \cdot \alpha = \sum_{i=-r}^s c_i \tau_*^i \cdot \alpha \in H_j(\widehat{\Omega}).$$

Here $\tau_* : H_j(\widehat{\Omega}) \rightarrow H_j(\widehat{\Omega})$ is the homology isomorphism induced by τ . Thus it defines the Λ -module $H_*(\widehat{\Omega})$ which is known as the Alexander invariant of the knot $K_\infty \subset S^n$. As a shorthand description of the first homology $H_1(\widehat{\Omega})$ of the infinite cyclic covering of the knot K_∞ complement $\Omega(G)$, one can also use its Alexander polynomial $\Delta_{K_\infty}(t)$, see [Ro, Ch. 7].

Lemma 4.2. *Let $\widehat{\Omega}$ and \widehat{P}_k , $k \geq 0$, be infinite cyclic coverings of $\Omega(G)$ and P_k in (4.4). Then we have a nested sequence*

$$\widehat{P}_0 \subset \widehat{P}_1 \subset \dots \subset \widehat{P}_k \subset \dots \widehat{\Omega}.$$

Proof. The nested sequence in (4.4) defines a sequence of monomorphisms of the fundamental groups as follows:

$$\pi_1(P_0) \hookrightarrow \pi_1(P_1) \hookrightarrow \dots \hookrightarrow \pi_1(P_k) \hookrightarrow \pi_1(P_{k+1}) \hookrightarrow \dots \hookrightarrow \pi_1(\Omega(G)).$$

Furthermore, we have the following commutative diagram:

$$\begin{array}{ccccccc} \dots \pi_1(P_{k-1}) & \xrightarrow{i_k} & \pi_1(P_k) & \xrightarrow{\quad} & \dots & \xrightarrow{i} & \pi_1(\Omega(G)) \\ \downarrow & & \downarrow & & & & \downarrow \\ \dots H_1(P_{k-1}) & \xrightarrow{i_{k*}} & H_1(P_k) & \xrightarrow{\quad} & \dots & \xrightarrow{i_*} & H_1(\Omega(G)) \\ \parallel & & \parallel & & & & \parallel \\ \dots \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \xrightarrow{\cong} & \dots & \xrightarrow{\cong} & \mathbb{Z} \end{array}$$

Here the vertical maps correspond to the Abelinization. So the lemma follows.

Lemma 4.3. $H_1(\widehat{\Omega}; \mathbb{Z}) \neq 0$.

Proof. Due to Lemma 4.2, we have excisive triads defined by $X = \widehat{P}_i$ and $Y = \widehat{\Omega} \setminus \widehat{P}_k$. So we have a Mayer-Vietoris exact sequence as follows:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_2(X \cap Y) & \longrightarrow & H_2(X) \oplus H_2(Y) & \longrightarrow & H_2(X \cup Y) \longrightarrow \\ & & \longrightarrow & & H_1(X \cap Y) & \longrightarrow & H_1(X) \oplus H_1(Y) \longrightarrow & H_1(X \cup Y) \longrightarrow \dots \end{array}$$

where $X \cap Y = \partial \widehat{P}_k$ and $X \cup Y = \widehat{\Omega}$.

For each $k \geq 0$, the boundary ∂P_k is homeomorphic to the product $S^{n-2} \times S^1$. It implies that $\partial \widehat{P}_k \approx S^{n-2} \times \mathbb{R}$, and hence $H_1(\partial \widehat{P}_k) = 0$. Therefore, the map

$$H_1(\widehat{P}_k) \oplus H_1(\widehat{\Omega} \setminus \widehat{P}_k) \longrightarrow H_1(\widehat{\Omega})$$

is injective.

On the other hand, $H_1(\widehat{P}_k)$ is not trivial due to the initial condition that the $(n-2)$ -knot $K \subset S^n$ is nontrivial. This completes the proof.

To finish the proof of Theorem 3.1, we observe that the $(n - 2)$ -knot $K_\infty = \Lambda(G) \subset S^n$ is nontrivial due to nontriviality of its Alexander invariant $H_*(\widehat{\Omega})$, which follows from Lemma 4.3. On the other hand, K_∞ is invariant for a non-elementary Kleinian group $G \subset \text{Möb}(n)$, and hence it is a wild knot due to [Ku]. The latter fact also follows from Lemma 4.3 and the additivity of the Alexander invariant $H_*(\widehat{\Omega})$ with respect to connected sum (4.5) of knots, see [L]. Obviously, any point $z \in K_\infty = \Lambda(G)$ that is the attractive fixed point of a loxodromic element $g \in G$ is a wild point of the knot K_∞ . The proof is completed by the well known fact (see [A1] for example) that such loxodromic fixed points are dense in the limit set $\Lambda(G) = K_\infty$.

5. QUASISYMMETRIC EMBEDDINGS AND VARIETY COMPONENTS

Let $\mathcal{QS}_{m,n}$, $n \geq m \geq 2$, be the space of conformal classes of quasimetric embeddings $S^m \hookrightarrow S^n$ with the compact-open topology. Up to Möbius transformations, such quasimetric embeddings $f : S^m \hookrightarrow S^n$ of m -sphere $S^m = \mathbb{R}^m \cup \infty$ can be represented by quasimetric embeddings of Euclidean m -space, $f : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$, which satisfy the following condition [TV]:

$$\frac{1}{\eta(\rho)} \leq \frac{|f(y) - f(x)|}{|f(z) - f(x)|} \leq \eta(\rho) \quad \text{if} \quad \frac{1}{\rho} \leq \frac{|y - x|}{|z - x|} \leq \rho, \quad \rho > 0,$$

where $\eta : [0, \infty) \rightarrow [0, \infty)$ some homeomorphism and $x, y, z \in \mathbb{R}^m$.

For a given lattice $\Gamma \subset O(m + 1, 1)$, we have a subspace $\mathcal{QS}_{m,n}(G) \subset \mathcal{QS}_{m,n}$ consisting of all Γ -equivariant quasimetric embeddings $f : S^m \hookrightarrow S^n$. Here the groups Γ and $f\Gamma f^{-1}$ act on $S^m = \partial H^{m+1}$ and $f(S^m) \subset S^n$ by restrictions of Möbius transformations in the corresponding spheres.

Since the canonical homeomorphisms of the limit sets in Tukia's [Tu] isomorphism theorem are in fact quasimetric, we immediately have an additional metric property of the knot K_∞ in Theorem 3.1:

Corollary 5.1. *For a given nontrivial ribbon m -knot $K \subset S^{m+2}$, $m \geq 2$, there is a quasimetric embedding $f : S^m \hookrightarrow S^{m+2}$ whose image is an everywhere wild m -knot $K_\infty = f(S^{m+2})$, infinitely compounded from K .*

Considering conjugations of the inclusion of a lattice $\Gamma \subset \text{SO}^\circ(m + 1, 1)$ to $\text{SO}^\circ(n + 1, 1)$ by quasiconformal self-homeomorphisms of \mathbb{H}^{n+1} compatible with the Γ -action, we have quasiconformal deformations of the inclusion $\Gamma \subset \text{SO}^\circ(n + 1, 1)$, i.e. curves in the varieties of representations $\mathcal{T}^n(\Gamma) \subset \mathcal{H}^{n+1}(\Gamma)$ corresponding to continuous families of such conjugations. Following to the classical terminology, we call such representations as quasi-Fuchsian ones. The Sullivan's stability theorem [Su, A9] implies that the set of conjugacy classes of quasi-Fuchsian representations is an open connected subspace of $\mathcal{T}^n(\Gamma) \subset \mathcal{H}^{n+1}(\Gamma)$. We denote the connected components of $\mathcal{T}^n(\Gamma)$ and $\mathcal{H}^{n+1}(\Gamma)$ containing this open subspace by $\mathcal{T}_\circ^n(\Gamma)$ and $\mathcal{H}_\circ^{n+1}(\Gamma)$.

Surprisingly, in contrast to the classical Teichmüller theory and the trivial space $\mathcal{H}^{m+1}(\Gamma)$, for $m \geq 2$, the varieties $\mathcal{T}^3(\Gamma)$ and $\mathcal{H}^4(\Gamma)$ may be disconnected for hyperbolic 3-lattices $\Gamma \subset \text{SO}^\circ(3, 1)$. Indeed, as it was pointed out by the author [A5], for some faithful discrete representations $\rho : \Gamma \rightarrow \text{SO}^\circ(4, 1)$, the limit set $\Lambda(\rho(\Gamma))$ may be an everywhere wild 2-sphere in S^3 (the boundary sphere of a wildly

embedded 3-ball). On the other hand, this is not a reason for $\mathcal{T}^4(\Gamma)$ and $\mathcal{H}^5(\Gamma)$ to be disconnected because the nerve of the constructed in [A5] knotting is 1-dimensional, so the limit 2-sphere $\Lambda(\rho\Gamma) \subset S^3$ is unknotted in S^4 .

Nevertheless, as another application of Theorem 3.1, we have:

Theorem 5.2. *Let $\Gamma \subset \mathrm{SO}^\circ(k, 1)$ be any uniform lattice from Theorem 3.1, and $k \geq 3$. Then the varieties $\mathcal{T}^{k+1}(\Gamma)$ and $\mathcal{H}^{k+2}(\Gamma)$ of conformal and hyperbolic structures on Γ (or equivalently, of conjugacy classes of discrete faithful representations $\Gamma \rightarrow \mathrm{SO}^\circ(k+2, 1)$) are disconnected.*

Proof. It immediately follows from Theorem 3.1 and the Sullivan's stability theorem because both (isomorphic) groups $\Gamma \subset \mathrm{Isom} \mathbb{H}^k \subset \mathrm{Isom} \mathbb{H}^{k+2}$ and $G = \rho\Gamma \subset \mathrm{Isom} \mathbb{H}^{k+2}$ constructed in Section 4 are convex cocompact (geometrically finite loxodromic groups). Namely, we see that the "knotted" representation ρ does not belong to the closure of the quasi-Fuchsian components $\mathcal{T}_\circ^{k+1}(\Gamma)$ and $\mathcal{H}_\circ^{k+2}(\Gamma)$ because, for any quasi-Fuchsian representation ρ' , the limit set $\Lambda(\rho'\Gamma)$ is an unknotted $(k-1)$ -sphere in S^{k+1} .

6. FIBER BUNDLES OVER HYPERBOLIC MANIFOLDS

We conclude our paper by pointing out a significant difference between geometric structures on fiber bundles over hyperbolic surfaces and m -manifolds, $m \geq 3$, which is based on Gluck's rigidity for Dehn surgery on high-dimensional knots (in contrast to Dehn surgery on classical 1-knots in S^3).

Given a closed hyperbolic m -manifold M with $\pi_1(M) = \Gamma \subset \mathrm{SO}^\circ(m, 1)$, each faithful discrete representation $\rho : \Gamma \rightarrow \mathrm{SO}^\circ(m+2)$ (a conjugacy class $[\rho] \in \mathcal{H}^{m+2}$) defines a $(m+2)$ -dimensional hyperbolic manifold $E = \mathbb{H}^{m+2}/\rho\Gamma$ and a closed conformal $(m+1)$ -manifold N at infinity of E , $N = \Omega(\rho\Gamma)/\rho\Gamma$, whose fundamental group satisfies the exact sequence (2.2) with $G = \rho\Gamma \subset \mathrm{SO}^\circ(m+2, 1)$. The manifolds E and N themselves are 2-plane and S^1 bundles over M provided the exterior of the $(m-1)$ -knot $\Lambda(\rho\Gamma) \subset S^{m+1}$ is homeomorphic to $B^m \times S^1$, that is the knot is trivial [Go]. Moreover, it turns out that (besides trivial bundles corresponding to quasi-Fuchsian representation of $\Gamma = \pi_1(M)$) there are such non-trivial circle and 2-plane bundles over M having conformal and hyperbolic structures and realized by the manifolds E and N , correspondingly. Namely, Gromov, Lawson and Thurston [GLT] have produced startling constructions of such representations ρ whose hyperbolic and conformal manifolds E and N are non-trivial 2-plane and circle bundles over a closed hyperbolic surface, see also [Ku]. Those fiber bundles are related to conformal realizations of Dehn surgeries on a classical 1-knot $S^1 \hookrightarrow S^3$. Similarly, existence of a hyperbolic structure on a 2-plane bundle over a hyperbolic m -manifold M (conformal structure on a circle bundle over M) whose holonomy group $\rho\Gamma$, $\Gamma = \pi_1(M)$, has an $(m-1)$ -knot in $S^{m+1} = \partial\mathbb{H}^{m+2}$ as the limit set implies a conformal realization of the corresponding Dehn surgery on the trivial knot $S^{m-1} \subset S^{m+1}$.

However, the Dehn surgery on high-dimensional knots $S^{m-1} \hookrightarrow S^{m+1}$, $m \geq 3$ is very rigid. This is related to the fact, for the first time observed by [Gl] in dimension $m = 3$ (see also [B, LS, Sw]), that two homeomorphisms of the boundary $\partial N(K) \approx S^{m-1} \times S^1$ of a regular neighborhood $N(K) \approx S^{m-1} \times B^2$ of a $(m-1)$ -knot $K \subset S^{m+1}$, $m \geq 3$, are pseudo-isotopic if and only if they are homotopic. The group of pseudo-isotopy classes of homeomorphisms of $S^{m-1} \times S^1$ is thus isomorphic to

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Here the first two factors correspond to orientation-reversals of S^{m-1} and S^1 respectively, and the third is generated by the following homeomorphism $\eta : S^{m-1} \times S^1 \rightarrow S^{m-1} \times S^1$,

$$\eta(x, \theta) = (\tau(\theta)(x), \theta); \quad x \in S^{m-1}, \quad \theta \in S^1, \quad (6.1)$$

where $\tau(\theta)$ is the rotation of the sphere S^{m-1} about its polar S^{m-3} through the angle θ .

Therefore, in contrast to the classical 1-knots in S^3 , each $(m-1)$ -knot $K \subset S^{m+1}$, $m \geq 3$, has the only one nontrivial Dehn surgery which is determined by the homeomorphism (6.1). This makes fiber bundles over closed hyperbolic m -manifolds, whose fibers are either 2-planes or circles and which have either hyperbolic structures (of infinite volume) or conformal structures, correspondingly, more rigid than the analogous fibrations over hyperbolic surfaces, see [GLT]. In particular it implies [A8]:

Theorem 6.1. *For a given closed hyperbolic m -manifold M , $m \geq 3$, there are at most two non-equivalent circle (or 2-plane) bundles over M allowing uniformizable conformal structures or complete hyperbolic metrics, respectively, with the development on a $(m-1)$ -knot complement.*

Finally we remark that there are at most finite number of equivalence classes of conformal $(m+1)$ -manifolds N (hyperbolic $(m+2)$ -manifolds E , $\partial E = N$) homotopy equivalent to a given closed hyperbolic m -manifold M whose developments are onto a (nontrivial) $(m-1)$ -knot complement. The number of such manifold equivalence classes depends of geometry of M , especially, of the number $b(M)$ of disjoint totally geodesic surfaces in M . In our example in Section 4 related to the (trefoil) 2-knot $K \subset S^4$, this number is more than 100 and, we think, it cannot be less than $C_m \cdot e$ where $C_m > 1$ is an universal constant and $e = e(K)$ is the potential energy of the m -knot K [BFHW, AS] (actually, for the used trefoil $k \subset S^3$, $e(k) \approx 74$).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OK 73019
E-mail address: `apanasov@ou.edu`

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